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CONTINUITY PROPERTIES OF MAJORITY RULE WITH INTERMEDIATE PREFER--ETC(U)

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CONTINUITY PROPERTIES OF

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INTERMEDIATE PREFERENCES

by

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and

Kuan-Pin Lin

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I. INTRODUCTION

Grandmont [7] recently provided a general possibility theorem for majority rule. The theorem assumes three general conditions on individual preferences and the distribution of these preferences in a society. These conditions are shown to imply that the majority rule relation is in the same family of individual relations as the domain of the distribution. The assumptions in [7] include (as special cases) single-peakedness, radial symmetry, the existence of a total median and special examples—each of which has been shown to be sufficient for the existence of a majority rule equilibrium. Related work on conditions which imply the existence of simple majority rule voting equilibria has been done by Plott [12], Sloss [14], McKelvey [9], McKelvey and Wendell [10] and McKelvey, Ordeshook and Ungar [11].

This paper shows that the assumptions in [7] implicitly contain a continuity property for the map from distributions of voter preferences to the index identified with the majority rule relation (Theorem 1). We then provide applications of this result to societies which satisfy the classical assumptions on preferences in the literature preceding [7] (viz. [1], [3], [4] and [16]), but which may have (atomic or nonatomic) measures of voters (as in [7]). In particular, Theorem 1 implies that the map from distributions of voters to majority rule equilibria is continuous in such societies (Corollaries 1 and 2). This gives answers to questions analogous to the ones raised in Denzau and Parks [5]. These continuity results are potentially useful for proving the existence of voting-market equilibria when societies satisfy the assumptions in [7] (as in [15] and [6] from the continuity properties in [5]).

In Section II, we present a model of majority rule with intermediate preferences in the sense of Grandmont. The reader is referred to [7] for detail. Section III states the main result of this paper: the continuity of majority rule. Applications of the above continuity property to majority rule equilibria are discussed for two classical types of individual preferences in Section IV. Finally, Section V contains proofs of the main theorem and its two corollaries.

II. GRANDMONT'S MODEL: NOTATION AND ASSUMPTIONS

Following [7]: X is a fixed set of alternatives on which each individual has a binary relation. $(R_a)_{a \in A}$ denotes a family of relations indexed by points a in an open convex subset A of E^n. This family is assumed to satisfy:

(H.1) (Weak Continuity): For every $x, y \in X$, the set $\{a \in A: xR_a y\}$ is closed in A.

(H.2) (Intermediate Preferences): For every $a', a'' \in A$, R_a is "between"[1] $R_{a'}$ and $R_{a''}$ whenever $a = \lambda \cdot a' + (1-\lambda) \cdot a''$ for $\lambda \in (0,1)$.

A society is specified by a probability measure ν on A. Let A′ and A″ be the intersections of A with two closed half-spaces determined by a hyperplane H. Every ν is assumed to satisfy:

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[1] R_a is said to be "between" $R_{a'}$ and $R_{a''}$ if for all $x, y \in X$, (i) $xR_{a'}y$ and $xR_{a''}y$ imply $xR_a y$; (ii) $xP_{a'}y$ and $xP_{a''}y$ imply $xP_a y$; (iii) ($xI_{a'}y$ and $xP_{a''}y$) or ($xP_{a'}y$ and $xI_{a''}y$) imply $xP_a y$.

(M. 1) There exists $a^* \in A$ such that for every hyperplane H of E^n, $\nu(A') = \nu(A'')$ if and only if $a^* \in H$.

N(A) denotes the collection of probability measures on A which satisfy (M. 1). The topology for N(A) is the relative topology induced by the topology of weak convergence on the collection of all probability measures on A (e.g., see [2]).

The majority rule relation, R_M, for any society $\nu \in N(A)$ is given by

$$xR_My \text{ if and only if } \nu\{a \in A: xR_ay\} \geq \nu\{a \in A: yR_ax\}.$$

III. CONTINUITY OF MAJORITY RULE

Grandmont [7] showed that (H. 1), (H. 2) and (M. 1) together imply that $R_M = R_{a^*}$ for some $a^* \in A$. These three conditions therefore define the "majority rule correspondence":

$$\phi(\nu) = \{a^* \in A: R_M = R_{a^*}\} \tag{1}$$

from each measure of voters, $\nu \in N(A)$, to the index(es) identified with the majority rule relation. In Lemma 1 (Section V) we show that ϕ is, in fact, a function. Therefore, we'll refer to ϕ as the "majority rule map."

We will prove the following:

Theorem 1: Suppose that every society, $\nu \in N(A)$, satisfies (H. 1), (H. 2) and (M. 1). Then the majority rule map defined by (1) is continuous.

IV. APPLICATIONS TO MAJORITY RULE EQUILIBRIA

In this section we apply Theorem 1 to societies in which preferences satisfy the classical assumptions in the papers preceding [7]. These applications provide results on the continuity properties of the correspondence from distributions of voters to voting equilibria (as in Denzau and Parks [5]).

For a majority relation R_M on the set of all possible alternatives X, a particular $x \in X$ is a majority rule equilibrium (or Condorcet winner) for the society ν if and only if xR_My for every $y \in X$. That is, x cannot be defeated by a majority in a pairwise vote against any other alternative.

The assumptions in Grandmont [7] do not assure that there is a majority rule equilibrium. Additionally, even when there are majority rule equilibria, they do not assure that there is any nicely behaved relation between the indices for the societies' majority rule relations and their maximal elements. However, in the following cases – which provided the basis for Grandmont's representation of preferences – the continuity of the correspondence from distributions of voters to majority rule equilibria is a straightforward consequence of Theorem 1.

(a) Quadratic Based Preferences

Davis, DeGroot and Hinich [4] and Tullock [16] assume that there is a Euclidean policy space, $X \subset E^n$, and that the preference relation of each individual, i, satisfies

$$xR_iy \text{ if and only if } \|x - x_i\| \leq \|y - x_i\| \tag{2}$$

for any $x, y \in X$ (where x_i is a unique "ideal point" and $\|\cdot\|$ is the usual Euclidean norm). (2) means that each individual ranks the possible policies according to their distance from his ideal point. Such preferences have been labelled "Type I preferences" (e.g., Kramer [8]).

They can be completely specified by letting the index for each preference relation be its ideal point, i.e., $a = x_i \in E^n$.

The above preferences have been generalized to "ellipsoidal" or "quadratic based" preferences (see, e.g., Riker and Ordeshook [13]). "Quadratic based preferences" for an individual, i, satisfy:

$$xR_iy \text{ if and only if } \|x - x_i\|_B \leq \|y - x_i\|_B \tag{3}$$

for any $x, y \in X$ (where x_i is a unique "ideal point" and $\|x\|_B = x' \cdot B \cdot x$ with B being a positive definite matrix. This means that the indifference contours of an individual are ellipsoids. The ratio of the major axis to the minor axis of an ellipsoidal indifference curve represents the relative salience of the dimensions. Given B, each preference relation can be completely specified by letting the index be its ideal point, i.e., $a = x_i \in E^n$.

Theorem 1 implies:

Corollary 1: <u>Suppose that every society, ν,</u>

1) <u>has quadratic based preferences, for a given</u> B, <u>on a Euclidean policy space which are indexed by their ideal points, and</u>
2) <u>satisfies</u> (M.1).

<u>Then the map from each society to its majority rule equilibrium is continuous</u>.

(b) <u>Single-Peaked Preferences</u>

Arrow [1] and Black [3] assume single-peakedness to assure the existence of a majority rule equilibrium. This requires that there is a strong ordering ($>_0$) on the alternative states so that the alternatives can be represented by a one-dimensional variable along which each individual's preference relation has a "single peak." When, additionally, X has a topology and the alternatives can be mapped by a homeomorphism, $\psi: X \to E^1$, which preserves the strong ordering $>_o$ (i.e., $\psi(x_1) > \psi(x_2)$ if and only if $x_1 >_o x_2$), we'll say that we have "real single-peakedness." This assumption includes the usual single-peakedness on a real-line policy space ($X \subset E^1$) as a special case. This will imply that, for each individual i, his preference relation R_i has two unique real elements, $c \leq d$, such that

(i) $\psi(x) < \psi(y) < c$ only if yP_ix

(ii) $\psi(x) < c \leq \psi(y) < d$ only if yP_ix

(iii) $c \leq \psi(x) \leq \psi(y) \leq d$ only if yI_ix

(iv) $c \leq \psi(x) \leq d < \psi(y)$ only if xP_iy

(v) $d \leq \psi(x) < \psi(y)$ only if xP_iy .

c and d are the left and right end points, respectively, of the single peak (or plateau) of the preference relation.

Each preference relation can be completely specified by the index $a = (c,d) \in E^2$ (e.g., Denzau and Parks [5], p. 855). When $c = d$ (i.e., there is a unique maximal element) for each individual, each preference relation can be specified by $a = c$ so that the index is one-dimensional (as in Grandmont [7]). Using the topology on X and the weak topology on N(A), Theorem 1 implies:

Corollary 2: <u>Suppose every society</u>, ν,

1) <u>has real single-peakedness with each preference relation indexed by</u> $a = (c,d) \in A \subset E^2$, <u>and</u>

2) <u>satisfies</u> (M. 1).

<u>Then the correspondence from each society to its majority rule equilibria</u>

is continuous.

The same is true when every society has real single-peaked preferences with unique peaks (i.e., $c = d$) which are indexed by $a = c \in E^1$.

V. PROOFS

In this section, we give proofs of the main theorem and its corollaries. We first develop properties of the majority rule map ϕ defined by (1), culminating in the result that Grandmont's conditions (H.1), (H.2) and (M.1) imply that ϕ is a continuous function.

For notational convenience, denote a hyperplane which contains $a \in E^n$ by $H(a)$. The disjoint open half-spaces determinated by this hyperplane will be denoted by $H^+(a)$ and $H^-(a)$. Their closures will have the usual notation of $\bar{H}^+(a)$ and $\bar{H}^-(a)$.

Lemma 1. The majority rule map ϕ: $N(A) \to A$ defined by $\phi(\nu) = \{a^* \in A: R_M = R_{a^*}\}$ for each $\nu \in N(A)$ is a (single-valued) function.

Proof. From Grandmont's main theorem ([7], p. 324), $\phi(\nu) = \{a^* \in A: \nu(A') = \nu(A'')$ for every hyperplane $H(a^*)$ in $E^n\} \neq \emptyset$ for each $\nu \in N(A)$. Suppose that there exist $a, b \in A$, $a \neq b$, where a and b are both a^*'s for the same $\nu \in N(A)$. Since $a \neq b$, there is a family $\mathcal{F}$ of parallel hyperplanes such that $a \in H(a) \in \mathcal{F}$ and $b \in H(b) \in \mathcal{F}$ while $H(a) \neq H(b)$. Since a is an a^*, $\nu(A') = \nu(A'')$ for H only if $a \in H$. Since $H(a) \neq H(b)$, we have $a \notin H(b)$. Therefore, $\nu(A') \neq \nu(A'')$ for $H(b)$.

But this contradicts b being an a*. Hence $\phi(\nu)$ is single-valued for each $\nu \in N(A)$. Q.E.D.

Lemma 2: Let $H(b)$ and $H(c)$ be from the same family $\mathcal{F}$ of parallel hyperplanes in E^n with $H^+(b) \cap H^-(c) \neq \emptyset$, then $a = \phi(\nu) \in H^+(b) \cap H^-(c)$ if and only if $\nu(H^+(b) \cap A) > \frac{1}{2}$ and $\nu(H^-(c) \cap A) > \frac{1}{2}$.

Proof: Suppose that $\nu(H^+(b) \cap A) > \frac{1}{2}$ and $\nu(H^-(c) \cap A) > \frac{1}{2}$, but $a = \phi(\nu) \notin H^+(b) \cap H^-(c)$. Then $\bar{H}^-(a) \subset E^n \setminus H^+(b)$ or $\bar{H}^+(a) \subset E^n \setminus H^-(c)$. Therefore, $\nu(\bar{H}^-(a) \cap A) \leqslant 1 - \nu(H^+(b) \cap A) < \frac{1}{2}$ or $\nu(\bar{H}^+(a) \cap A) \leqslant 1 - \nu(H^-(c) \cap A) < \frac{1}{2}$. But (M.1) implies that $\nu(\bar{H}^+(a) \cap A) = \nu(\bar{H}^-(a) \cap A) \geqslant \frac{1}{2}$. A contradiction.

To show the converse, let $a \in H^+(b)$, then $\bar{H}^+(a) \subset H^+(b)$. So $\nu(\bar{H}^+(a) \cap A) \leqslant \nu(H^+(b) \cap A)$. But (M.1) implies $\nu(\bar{H}^+(a) \cap A) \geqslant \frac{1}{2}$. Therefore, $\nu(H^+(b) \cap A) \geqslant \frac{1}{2}$. Suppose $\nu(H^+(b) \cap A) = \frac{1}{2}$. Write $H^+(b) = [H^+(b) \cap H^-(a)] \cup \bar{H}^+(a)$, then $\frac{1}{2} = \nu(H^+(b) \cap A) = \nu(H^+(b) \cap H^-(a) \cap A) + \nu(\bar{H}^+(a) \cap A)$. But $\nu(\bar{H}^+(a) \cap A) \geqslant \frac{1}{2}$ by (M.1). Therefore, $\nu(H^+(b) \cap H^-(a) \cap A) = 0$. Since A is open and convex, then $\nu(H^+(b) \cap H^-(a) \cap A) = 0$ says that there is some $d \in H^+(b) \cap H^-(a)$ with $\nu(A') = \nu(A'')$ for $H(d) \in \mathcal{F}$. But then $\nu(A') = \nu(A'')$ does not occur only if $a \in H(a)$, which contradicts (M.1). Hence $\nu(H^+(b) \cap A) > \frac{1}{2}$. A similar argument establishes $\nu(H^-(c) \cap A) > \frac{1}{2}$ (see Figure 1 for the graphical interpretation of the proof for the case $a \in H^+(b)$ in E^2).

Q.E.D.

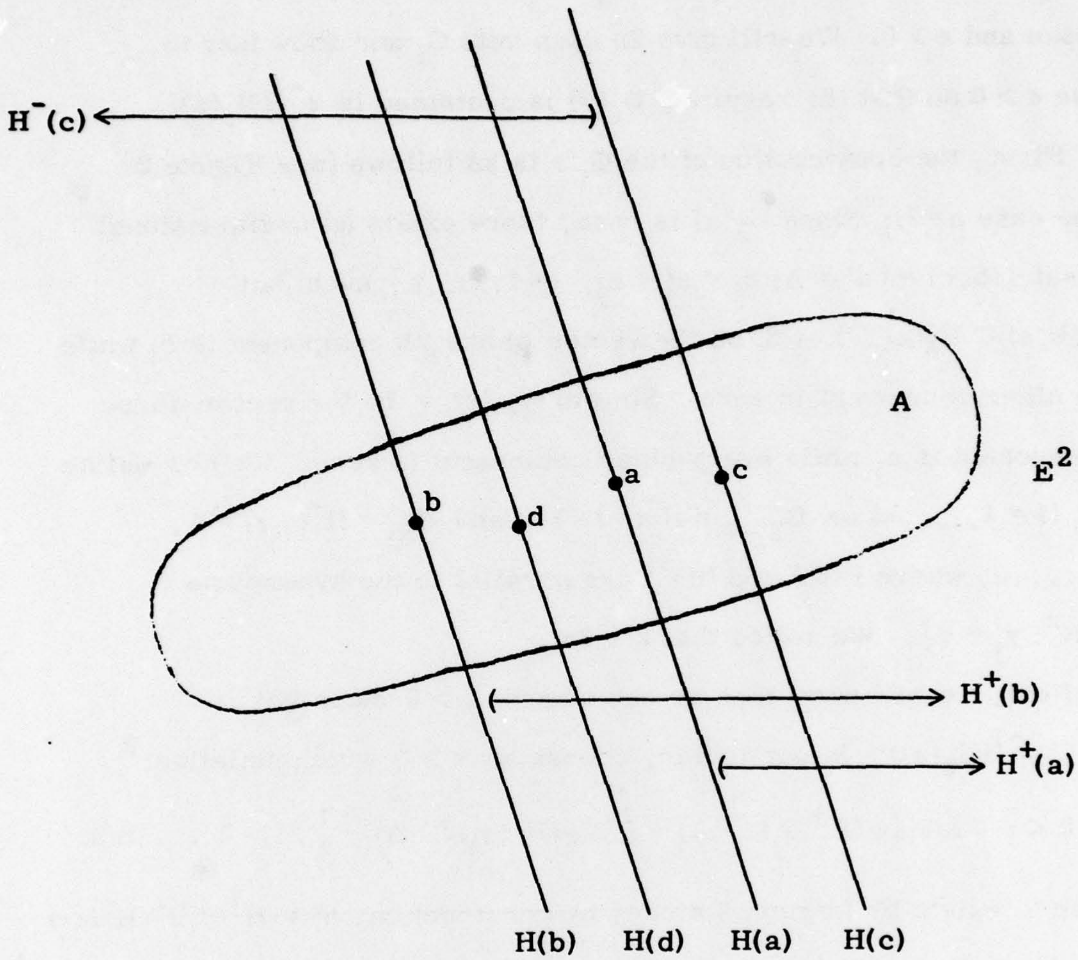

Figure 1

We are now in a position to prove our main theorem about the continuity of the majority rule map with intermediate preferences.

<u>Proof of Theorem 1</u>: First, by Lemma 1, ϕ is a single-valued function. Therefore we need to prove that for any $a \in A$, $\phi^{-1}(U_\delta(a))$ is open in N(A), where $U_\delta(a) = \{a' \in A: \|a - a'\| < \delta\}$ with $\delta > 0$ and $\|\cdot\|$ the Euclidean norm defined on E^n. Choose <u>any</u> $\nu \in \phi^{-1}(U_\delta(a))$. A neighborhood basis of ν is given by the

sets $B_\epsilon(\nu) = \{\nu' \in N(A): \nu'(G_i) > \nu(G_i) - \epsilon,\ i = 1,\ldots,k\}$ where the G_i are open and $\epsilon > 0$. We will give 2n open sets G_i and show how to choose $\epsilon > 0$ so that the resulting $B_\epsilon(\nu)$ is contained in $\phi^{-1}(U_\delta(a))$.

First, the construction of the G_i's is as follows (see Figure 2 for the case $n = 2$): Since $U_\delta(a)$ is open, there exists an n-dimensional open set $I(b,c) = \{a' \in A: b_j < a'_j < c_j,\ j = 1,\ldots,n\}$ such that $a \in I(b,c) \subset U_\delta(a)$. Let β_j be the vector whose jth component is b_j while every other component is zero. Similarly, let γ_j be the vector whose jth component is c_j while every other component is zero. We now define the G_i $(i = 1,\ldots,k)$ as $G_{2j-1} = H^+(\beta_j) \cap A$ and $G_{2j} = H^-(\gamma_j) \cap A$, $j = 1,\ldots,n$, where $H(\beta_j)$ and $H(\gamma_j)$ are parallel to the hyperplane $\{y \in E^n: y_j = 0\}$. We notice that $k = 2n$.

Next, we will show that we can choose $\epsilon > 0$ such that $B_\epsilon(\nu) \subset \phi^{-1}(U_\delta(a))$. In particular, choose an $\epsilon > 0$ which satisfies:

$$0 < \epsilon < \min\{\nu(H^+(\beta_j) \cap A) - \tfrac{1}{2},\ \nu(H^-(\gamma_j) \cap A) - \tfrac{1}{2},\ j = 1,\ldots,n\}.$$

Such an ϵ exists by Lemma 2 since, by construction, $a' \in H^+(\beta_j) \cap H^-(r_j)$ for each $j = 1,\ldots,n$. Now, by definition, for any $\nu' \in B_\epsilon(\nu)$, we have $\nu'(H^+(\beta_j) \cap A) > \nu(H^+(\beta_j) \cap A) - \epsilon$ and $\nu'(H^-(\gamma_j) \cap A) > \nu(H^-(\gamma_j) \cap A) - \epsilon$ for $j = 1,\ldots,n$. From the chosen ϵ, then, $\nu'(H^+(\beta_j) \cap A) > \frac{1}{2}$ and $\nu'(H^-(\gamma_j) \cap A) > \frac{1}{2}$. By Lemma 2, the above inequalities imply that

$$d = \phi(\nu') \in H^+(\beta_j) \cap H^-(\gamma_j),$$

i.e., $b_j < d_j < c_j$. Therefore, $d \in I(b,c) \subset U_\delta(a)$. Hence $\nu' \in \phi^{-1}(U_\delta(a))$. This shows that $B_\epsilon(\nu) \subset \phi^{-1}(U_\delta(a))$. This means that every $\nu \in \phi^{-1}(U_\delta(a))$ is an interior point, so $\phi^{-1}(U_\delta(a))$ is open.

Q.E.D.

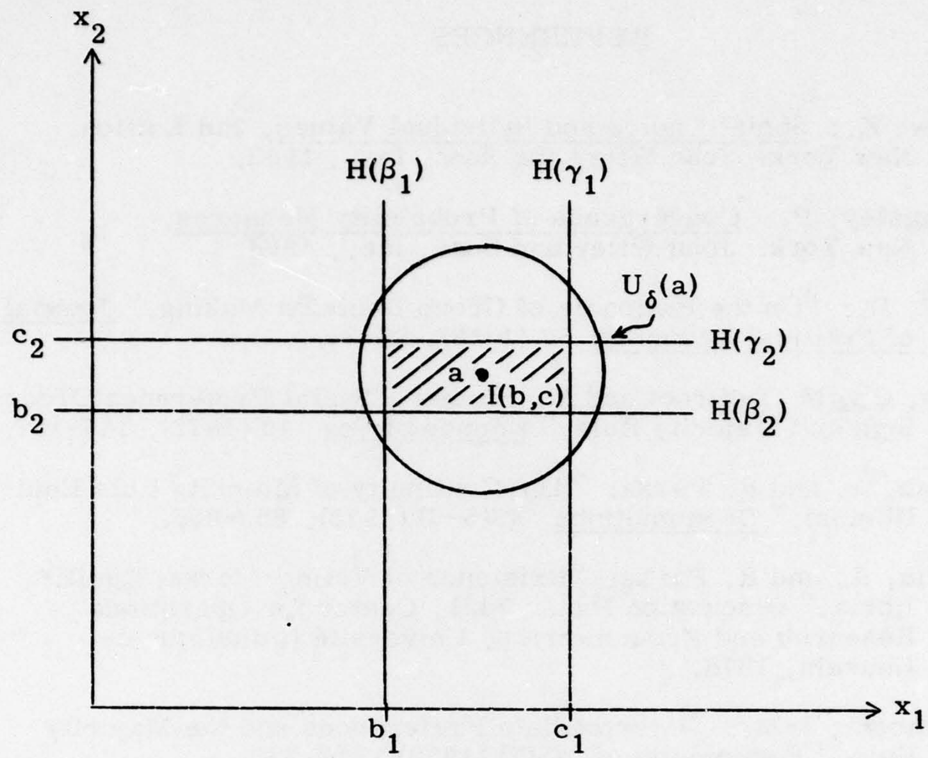

Figure 2

Proof of Corollary 1: Quadratic based preferences on a Euclidean policy space indexed by their ideal points satisfy (H. 1) and (H. 2). Therefore, ϕ: N(A) → A is continuous. Since, for $R_{a*} = R_M$, the index a* is the unique majority rule equilibrium, the corollary follows. Q.E.D.

Proof of Corollary 2: Real single-peakedness indexed by a = (c,d) for the society means that the preferences satisfy (H. 1) and (H. 2). The set of majority rule equilibria for a particular $\nu \in N$ is $\{x \in X: \psi(x) \in [c,d]\}$, where [c,d] is the closed interval whose endpoints are given by $\phi(\nu)$. Since ψ is a homeomorphism, ψ^{-1} is continuous. Therefore, since the closed interval [c,d] is a continuous correspondence of its endpoints and ϕ is continuous by Theorem 1, the corollary follows.

Q.E.D.

REFERENCES

[1] Arrow, K.: Social Choice and Individual Values, 2nd Edition. New York: John Wiley and Sons, Inc., 1963.

[2] Billingsley, P.: Convergence of Probability Measures. New York: John Wiley and Sons, Inc., 1968.

[3] Black, D.: "On the Rationale of Group Decision Making," Journal of Political Economy 56 (1948), 23-34.

[4] Davis, O., M. DeGroot and M. Hinich: "Social Preference Orderings and Majority Rule," Econometrica 40 (1972), 147-157.

[5] Denzau, A. and R. Parks: "The Continuity of Majority Rule Equilibrium," Econometrica 43(5-6) (1975), 853-866.

[6] Denzau, A. and R. Parks: "Existence of Voting-Market Equilibria," Discussion Paper 7831, Center for Operations Research and Econometrics, Université Catholique de Louvain, 1978.

[7] Grandmont, J-M.: "Intermediate Preferences and the Majority Rule," Econometrica 46(2) (1978), 317-330.

[8] Kramer, G.: "A Dynamical Model of Political Equilibrium," Journal of Economic Theory 16 (1977), 310-334.

[9] McKelvey, R., "Policy Related Voting and Electoral Equilibrium," Econometrica 43 (5-6) (1975), 815-844.

[10] McKelvey, R. and R. Wendell, "Voting Equilibria in Multidimensional Choice Spaces," Mathematics of Operations Research 1 (2) (1976), 144-158.

[11] McKelvey, R., P. Ordeshook and P. Ungar, "Conditions for Voting Equilibria in Continuous Voter Distributions," Social Science Working Paper Number 231 (1978), California Institute of Technology.

[12] Plott, C., "A Notion of Equilibrium and Its Possibility Under Majority Rule," American Economic Review 57 (1967), 787-806.

[13] Riker, W. and P. Ordeshook: An Introduction to Positive Political Theory. Englewood Cliffs: Prentice-Hall, 1973.

[14] Sloss, J., "Stable Outcomes in Majority Voting Games," Public Choice 15 (1973), 19-48.

[15] Slutsky, S.: "A Voting Model for the Allocation of Public Goods: Existence of an Equilibrium," Journal of Economic Theory 14 (1977), 299-325.

[16] Tullock, G.: "The General Irrelevance of the General Impossibility Theorem," Quarterly Journal of Economics 81 (1967), 256-270.

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13. ABSTRACT

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